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Altering structures to clear vibration frequencies from a band

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Abstract

Structures vibrate, and sometimes with frequencies that are not wanted. Structures are commonly required to be resonance-free within certain frequency ranges, and if they do have undesirable frequencies, these can be moved by changing the structure stiffness, or mass, or both. A mixed stiffness/flexibility formulation of the vibration problem presents alternative condensations to stiffness and flexibility eigenvalue equations for an altered structure. The flexibility form gives more compact equations, and this is developed to solve a parent problem where a structure has a single frequency in a nominated band, to be removed by adjusting the stiffness of a brace stressing the structure in a single way. Interestingly, if the original eigenvalue problem has a Sturm sequence, the frequency exclusion problem can be solved without determining any frequency or mode of the original structure. © 2001 Elsevier Science Ltd. All rights reserved.

Keywords: Bracing; Vibration frequency; Eigenvalue problem; Frequency exclusion

1. Introduction

Structures have natural frequencies of vibration, and some of them may be undesirable. Any unwanted frequencies of an existing structure were probably eliminated in the original design, but we may be concerned with a proposed structure, or we could be looking at an existing structure that has been changed in some way, either intentionally or through damage, and now has frequencies that are not wanted.

Stiffness equations for free vibration of a structure have the eigenvalue form

$$\left(\underset{\sim}{\mathbf{K}} - \omega^2 \underset{\sim}{\mathbf{M}}\right)\underset{\sim}{\mathbf{u}} = \underset{\sim}{\mathbf{0}}, \tag{1}$$

where $\underset{\sim}{\mathbf{K}}$ and $\underset{\sim}{\mathbf{M}}$ are the structure stiffness and mass matrices, respectively. Solutions are associated pairs $(\omega, \underset{\sim}{\mathbf{u}})$ which are a vibration frequency and vibration mode respectively.

Changes to the stiffness, here called *bracing*, or to the mass, will change these solutions. This paper concentrates on changes to the stiffness, but as Eq. (1) has the equivalent form,

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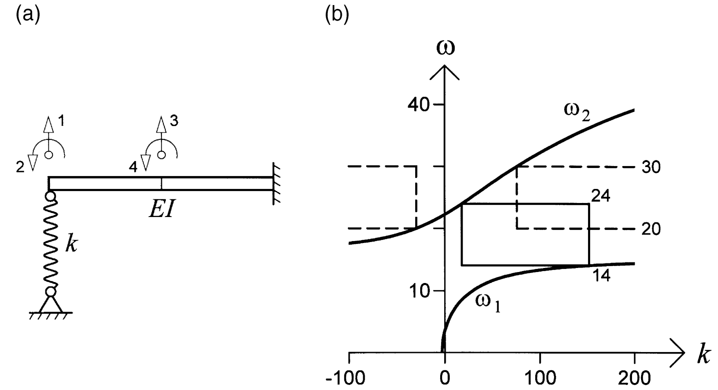

Fig. 1. Braced cantilever: (a) model and (b) bracing curves.

$$\left(\underset{\sim}{\mathbf{M}} - 1/\omega^2\underset{\sim}{\mathbf{K}}\right)\underset{\sim}{\mathbf{u}} = \underset{\sim}{\mathbf{0}}, \tag{1a}$$

changes to the mass are fundamentally the same, although a couple of differences in detail are noted in the closing remarks.

As an example of bracing, consider the cantilever beam, modelled with two finite elements, shown in Fig. 1a. The model has four freedoms, and gives vibration frequencies of 3.518, 22.22, 75.16 and 218.1 (these are values of the dimensionless frequency $\omega L^2\sqrt{\rho/EI}$, where L is the span, EI, the flexural rigidity, and ρ, the mass per unit length).

When braced, here by a simple spring supporting the free end, all frequencies are raised by an amount depending on the stiffness of the brace. The relationships between the vibration frequencies and brace stiffness are *bracing curves*, and the curves for the first two modes of the cantilever, braced at the tip, are shown in Fig. 1b. The figure is plotted in axes of k and ω, rather than in the eigenvalues k and ω^2. This has been done for compactness, and is continued throughout the paper. Nothing essential is lost in this.

If we wished to have this structure with no frequencies in the range of 14–24, then from an inspection of the solid rectangle of Fig. 1b, this is realised with any spring with stiffness in the range of the rectangle, 18–152 (these are values of the dimensionless parameter kL^3/EI, and how these are calculated directly will be shown later). Similarly, the dashed rectangle shows that all frequencies between 20 and 30 are removed if the spring stiffness exceeds 76. Bracing with negative stiffness makes no physical sense in this structure, but in others it may simply be reducing the stiffness of an existing element. There is no theoretical reason to discount negative bracing, and a brace with stiffness less than −29 will equally exclude all frequencies in the same range of 20–30. These rectangles, bounded by the relevant frequency range, are completely free of bracing curves.

No tip brace can exclude all frequencies in the range of 10–30, and we cannot draw a rectangle, bounded by frequencies of 10 and 30 and free of curves although it will be shown that we can construct a brace to produce this result.

Bracing to clear frequency bands is equivalent to locating open spaces on the bracing diagram, or if the form of the brace is unspecified, to constructing a bracing diagram with the open space in the right place.

2. Formulation

Free vibration equations of the original structure are

$$\left(\underset{\sim}{\mathbf{K}} - \omega^2\underset{\sim}{\mathbf{M}}\right)\underset{\sim}{\mathbf{u}} = \underset{\sim}{\mathbf{0}}.$$

Bracing will add stiffness to some displacements $\mathbf{u}_{\mathrm{b}}$, related to the structure displacements by

$$\mathbf{u}_{\mathrm{b}} = \mathbf{G}\mathbf{u}, \tag{2}$$

where $\mathbf{G}$ gives the geometric connections of the brace.

The bracing will have forces $\mathbf{N}_{\mathrm{b}}$ in the vibration mode, related to the braced displacements by

$$\mathbf{u}_{\mathrm{b}} = \mathbf{F}\mathbf{N}_{\mathrm{b}}. \tag{3}$$

Contragredience gives these forces in the original coordinates as

$$\mathbf{P}_{\mathrm{b}} = \mathbf{G}^{\mathrm{T}}\mathbf{N}_{\mathrm{b}} \tag{4}$$

and with the total of the internal forces applied to the nodes now $\left(\mathbf{K} - \omega^2\mathbf{M}\right)\mathbf{u} + \mathbf{P}_{\mathrm{b}}$, the equilibrium equations of the braced structure are

$$\left(\mathbf{K} - \omega^2\mathbf{M}\right)\mathbf{u} + \mathbf{G}^{\mathrm{T}}\mathbf{N}_{\mathrm{b}} = \mathbf{0}. \tag{5}$$

Eqs. (2), (3) and (5) combine to give

$$\begin{bmatrix} \mathbf{K} - \omega^2\mathbf{M} & \mathbf{G}^{\mathrm{T}} \\ \mathbf{G} & -\mathbf{F} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{N}_{\mathrm{b}} \end{bmatrix} = \mathbf{0}. \tag{6}$$

Eq. (6) describes the new, braced structure in a mixed formulation. The original structure is included through the original, unaltered, stiffness matrix occupying the leading diagonal submatrix position, and the bracing is included through a flexibility relation augmenting the original.

Static condensation to $\mathbf{u}$ eliminates $\mathbf{N}_{\mathrm{b}}$ and gives the assembled stiffness of the braced structure as

$$\left(\mathbf{K} + \mathbf{G}^{\mathrm{T}}\mathbf{F}^{-1}\mathbf{G} - \omega^2\mathbf{M}\right)\mathbf{u} = \mathbf{0} \tag{7}$$

(Williams and Anderson, 1983) which is a new eigenvalue problem with the same size as the original.

Alternatively, condensation to $\mathbf{N}_{\mathrm{b}}$ eliminates $\mathbf{u}$, giving

$$\left(\mathbf{G}\left(\mathbf{K} - \omega^2\mathbf{M}\right)^{-1}\mathbf{G}^{\mathrm{T}} + \mathbf{F}\right)\mathbf{N}_{\mathrm{b}} = \mathbf{0}, \tag{8}$$

which is also an eigenvalue problem, but with the size of $\mathbf{N}_{\mathrm{b}}$, the number of independent stressing modes of the brace, here called the *rank* of the brace (it is the same as the rank of the brace stiffness matrix $\mathbf{G}^{\mathrm{T}}\mathbf{F}^{-1}\mathbf{G}$). This is likely to be a much smaller problem than that of Eq. (7).

Each form has its applications. Returning to the example of Fig. 1, if we were to ask, 'what are the frequencies of a cantilever when braced at the tip with a stiffness of 100?' we know $\mathbf{F}^{-1}$ (1 0 0) and we know $\mathbf{G}$ (its a brace of the tip deflection, freedom 1), and Eq. (7) will give the required frequencies. Diagrammatically, we are drawing a vertical line at the $k = 100$ mark of Fig. 1b, and searching along it for solutions. It is not different from Eq. (1), which searches the $k = 0$ line.

Had we asked, 'what stiffness of tip brace will give a frequency of 30?' Eq. (8) is more attractive than Eq. (7) because it is likely to be much smaller. This draws a horizontal line at $\omega = 30$ and scans it for solutions.

A more open and interesting question is, 'If the second natural frequency is to be 150, what rank of brace is required, and how should it be connected?'

2.1. Rank 1 bracing

Rank 1 bracing is the simplest. With a single stress mode, $\underset{\sim}{\mathbf{N}}_b$ becomes a scalar N_b, and $\underset{\sim}{\mathbf{F}}$ similarly becomes f. The connections $\underset{\sim}{\mathbf{G}}$ are a vector $\underset{\sim}{\mathbf{g}}$, and Eq. (8) is now

$$\left(\underset{\sim}{\mathbf{g}}\underset{\sim}{\mathbf{K}}(\omega)^{-1}\underset{\sim}{\mathbf{g}}^{\mathrm{T}} + f\right)N_b = 0, \tag{8a}$$

where $\underset{\sim}{\mathbf{K}}(\omega)$ has been written for $\left(\underset{\sim}{\mathbf{K}} - \omega^2\underset{\sim}{\mathbf{M}}\right)$. This single eigenvalue equation has the solution

$$f = 1/k = -\underset{\sim}{\mathbf{g}}\underset{\sim}{\mathbf{K}}(\omega)^{-1}\underset{\sim}{\mathbf{g}}^{\mathrm{T}}. \tag{9}$$

An added support is a typical example of a rank 1 brace, as is the simple spring or truss member added to the cantilever of Fig. 1. Freedom 1 is the only one braced here, and the connection for this is $\underset{\sim}{\mathbf{g}} = [1 \quad 0 \quad 0 \quad 0]$. To calculate the spring stiffness which gives a frequency of 30, $\underset{\sim}{\mathbf{K}}(30)$ is formed and $k = 75.8$ is calculated from Eq. (9).

Brandon (1984) has considered the analysis of altered structures, concentrating on properties of the receptance (flexibility) matrix after the changes. The above analysis adds nothing to his results, except, perhaps, a different perspective.

While rank 1 bracing is the simplest, it is also completely general in that bracing of any rank r can be written as r independently applied rank 1 braces, as noted by Brandon. The equations take this form if we calculate any diagonal factorisation of $\underset{\sim}{\mathbf{F}}$ (Gauss or Choleski would do), and use it to transform the stresses into a natural system.

2.2. Shapes of the bracing curves

Except when ω is a natural frequency ω_i, $\underset{\sim}{\mathbf{K}}(\omega)$ is invertible and Eq. (9) has one solution only. Any horizontal line drawn in Fig. 1b will intersect exactly one bracing curve, with a limiting exception when $f = 0$, which is $k = \pm\infty$. The figure is divided into horizontal layers, each containing a single bracing curve, which must be monotonic increasing, as shown in the appendix. The curves asymptote to those values of ω which give $f = 0$, and are 'S' shaped in between (except for the lowest and uppermost curves, which have only a single asymptote).

The case of a singular $\underset{\sim}{\mathbf{K}}(\omega_i)$ is discussed in the appendix, where it is shown that nothing essential in the above description is changed.

This graph structure, with each curve in its own horizontal region, shown in Fig. 2a, is a property of rank 1 bracing. For rank r, Eq. (8) will have r solutions, and a horizontal line will cut r bracing curves. These 'S' shaped bracing curves are the 'auxiliary function' used by Weissenburger (1968) and Brandon (1984, 1990), although these authors use a flexibility axis where stiffness is used here. A hybrid view gives an interesting alterative interpretation. Plot the curves using stiffnesses in the range of -1 to $+1$ (Fig. 3a), with a similar plot for flexibilities ranging from $+1$ to -1 (Fig. 3b). The limits, of course, are the same, and the two plots can be joined smoothly, [1] at either boundary (Fig. 3c). If the joined plot is wrapped into a cylinder the bracing curves now appear as the single helix of Fig. 3d. The asymptotes in the stiffness plot are the intercepts of the Weissenburger–Brandon flexibility plot, and equally, the asymptotes of the flexibility plot are the intercepts in the stiffness plot, which are the natural frequencies.

[1] The function will not be analytic at the join, but it will be C^1 continuous in all cases, except for rare circumstances when the continuity is reduced to C^0. Generally, it will appear to be fairly smooth.

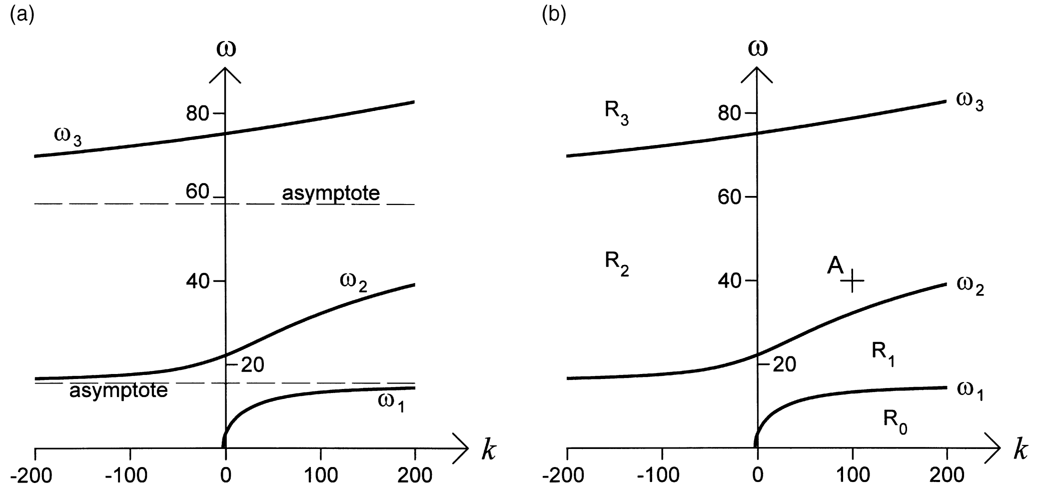

Fig. 2. Bracing curves (a) contained in horizontal layers between asymptotes, and (b) as boundaries of mode count regions.

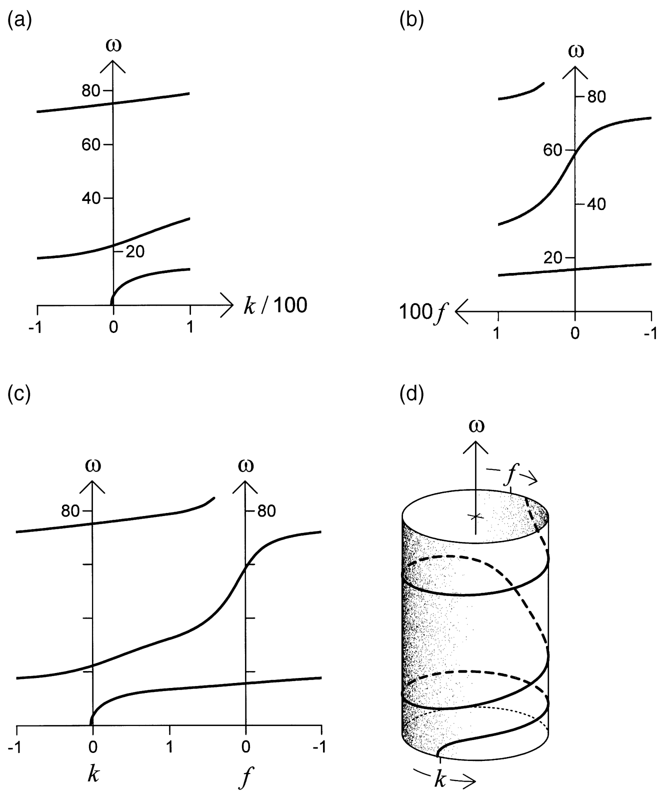

Fig. 3. Bracing curves (a) for stiffnesses in the range −1 to +1, (b) for flexibilities in the range +1 to −1, (c) for both, and (d) wrapped onto a cylinder.

Rank 1 bracing is a single helix winding its way up a cylinder, the circumference of which is marked out partly in stiffness, and partly in flexibility. Rank 2 bracing is simply two helices on the cylinder. [2]

2.3. Mode counting

The eigenvalue problem of Eq. (1) has symmetric $\mathbf{K}$ and $\mathbf{M}$, and if we are considering a structure, as distinct from a mechanism, $\mathbf{K}$ is positive definite. This eigenvalue problem can be solved by the Sturm sequence or mode count algorithm (Parlett, 1980; Wittrick and Williams, 1971). The bracing curves can be seen to divide the k–ω plane into regions; that below the ω_1 curve, that between the ω_1 and ω_2 curves, and so on. These are designated R_0, R_1, R_2, etc., with the number obviously indicating the lower of the bounding curves, as shown in Fig. 2b. A mode counting algorithm gives this region number as a function of k and ω.

Eq. (7) has a particularly simple mode count algorithm. $\mathbf{K}(k,\omega) = \mathbf{K} + \mathbf{G}^{\mathrm{T}}\mathbf{F}^{-1}\mathbf{G} - \omega^2\mathbf{M}$ is formed at the desired point, (k,ω), and is converted to a diagonal form by any real congruent transform whatever (Gauss factorisation, $\mathbf{K} = \mathbf{L}\mathbf{D}\mathbf{L}^{\mathrm{T}}$, is a popular method). The mode count is the number of negative elements of this diagonal form. As an example, the cantilever, with a brace stiffness of 100 and a frequency of 40, has

$$\mathbf{K}(100,40) = \begin{bmatrix} -201.1 & 3.048 & -198.9 & 36.38 \\ 3.048 & 6.095 & -36.38 & 5.429 \\ -198.9 & -36.38 & -402.3 & 0 \\ 36.38 & 5.249 & 0 & 12.19 \end{bmatrix}, \tag{10}$$

giving the Gauss factors

$$\mathbf{L} = \begin{bmatrix} 1 & . & . & . \\ -0.0152 & 1 & . & . \\ 0.9886 & -6.414 & 1 & . \\ -0.1809 & 0.9737 & -0.0052 & 1 \end{bmatrix}, \qquad \mathbf{D} = \begin{bmatrix} -201.1 & & & \\ & 6.141 & & \\ & & -458.4 & \\ & & & 12.96 \end{bmatrix}. \tag{11}$$

$\mathbf{D}$ has two negative elements, indicating that the point $(k,\omega) = (100,40)$ lies in region R_2, as is shown by point A in Fig. 2b.

The mode count ranges from 0 potentially to N, the number of freedoms in the problem. Eq. (8) has a dimension of r (here 1), so clearly needs a different mode count algorithm (Williams and Anderson, 1983; Lawther, 1995), which, in passing, provides an alternative proof for the monotonic slope of the bracing curves.

The brace stiffness calculation of Eq. (9), when combined with monotonic slope and mode counting, determines exactly which mode has been braced. Forming, factoring and mode counting $\mathbf{K}(\omega) = \mathbf{K} - \omega^2\mathbf{M}$ determines the mode count for a point $(k = 0,\omega)$ on the ω axis. Eq. (9) gives the bracing stiffness needed for this value of ω to become a natural frequency. If it is positive, the bracing curve must be to the right of $(0,\omega)$ and is therefore on the lower boundary of the region containing $(0,\omega)$. The mode braced is equal to the mode count calculated. On the other hand, a negative bracing stiffness leads left, to the upper boundary of the region, and the mode which is *mode count* $+1$. In Fig. 4, points A and B are on the ω axis in region R_1. Bracing at ω_A involves positive stiffness, and leads to A′ on the lower bound ω_1 curve. Bracing at ω_B requires negative stiffness, leading to B′ on the upper bound curve ω_2. The significant difference between points A and B is that they are on opposite sides of the asymptote separating the two bracing curves which bound the R_1 mode count region.

[2] This assumes that the rank 2 brace is described by a single stiffness parameter.

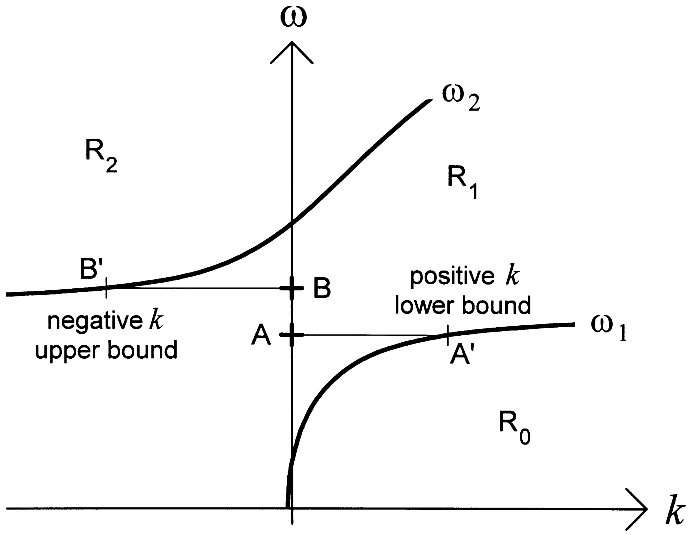

Fig. 4. Bracing to the upper and lower bound curves.

3. Excluding frequencies

Adding a brace of rank r to a structure changes the frequencies. If the brace has positive stiffness, the changes cannot be negative. The nth frequency ω_n is raised by at least 0, and at most up to ω_{n+r}. This is well known. As a consequence, if a structure has r frequencies in the range $[\omega_L, \omega_U]$ then removing them will require r braces of rank 1, or a rank r brace, at minimum. The parent problem of the rank 1 brace is therefore concerned with excluding precisely one frequency. Any fewer and there is no exclusion problem; any more and a higher rank brace is needed.

The problem is now stated as,

> Given a structure which is known to have precisely 1 natural frequency between a lower bound frequency ω_L and an upper bound frequency ω_U, can a rank 1 brace be designed to remove the frequency from the range $[\omega_L, \omega_U]$?

Let the included frequency be the nth, the bracing stiffnesses required to make ω_L a natural frequency be k_L, and that to brace to the upper frequency be k_U. The point, $L = [k_L, \omega_L]$, is on either the $(n-1)$th bracing curve, or the nth, depending on the sign of k_L, and $U = [k_U, \omega_U]$ is similarly on the nth or the $(n+1)$th. U and L are either

(i) both on the same curve, the nth,
(ii) on adjacent curves ($n-1$ and n, or n and $n+1$) or
(iii) are on the $(n-1)$th and $(n+1)$th, separated by the nth curve,
which come about when
(i) k_L is negative and k_U is positive,
(ii) k_L and k_U have the same sign,
(iii) k_L is positive and k_U is negative, respectively.

These possibilities are shown in the *exclusion diagrams* of Fig. 5.

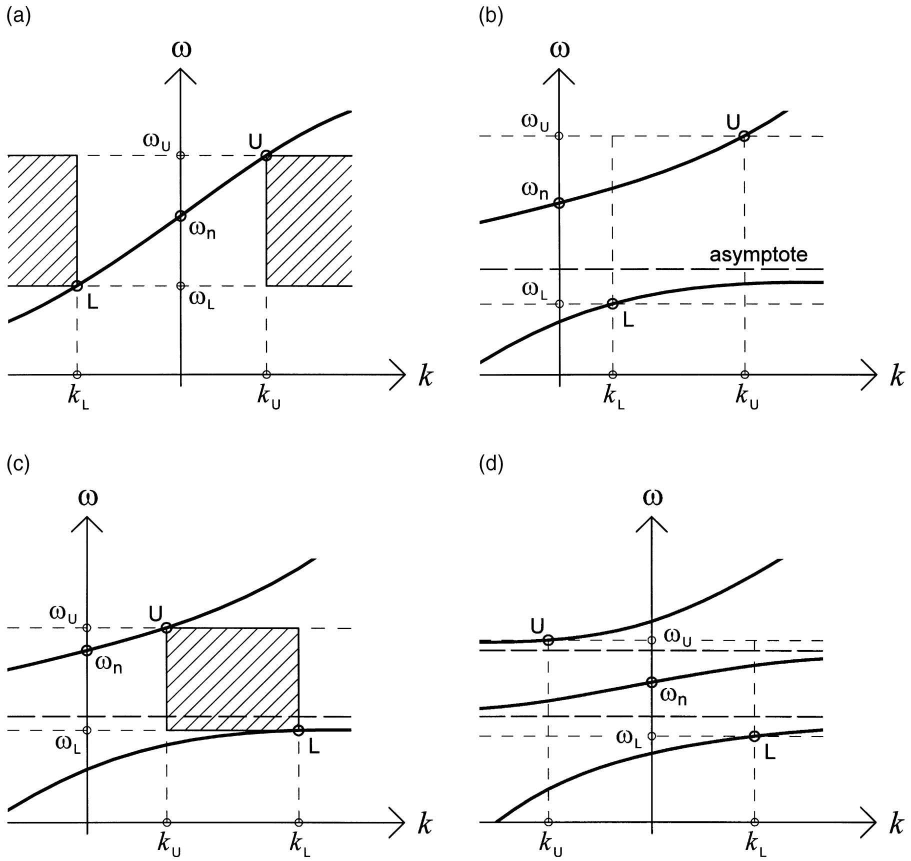

Fig. 5. Exclusion diagram possibilities. Note: these possibilities are labelled (a)–(d).

Point L is necessarily below U, and in cases (i) and (iii), their relative horizontal positions are determined, but not in (ii), so four different exclusion diagrams are possible. Attention is drawn to three rectangular regions of the diagrams, all bounded vertically by $\omega = \omega_{\mathrm{L}}$ and $\omega = \omega_{\mathrm{U}}$, and with horizontal ranges from $k = -\infty$ to the lesser of k_{L} and k_{U}, from the lesser to the greater of these, and from the greater to $k = +\infty$.

In Fig. 5a, where L and U lie on the same bracing curve, we must have $k_{\mathrm{L}} < k_{\mathrm{U}}$. The single bracing curve passes diagonally through the centre rectangle. With rank 1 bracing, any horizontal line drawn on the bracing diagram will intersect precisely one bracing curve, and all horizontal lines between ω_{L} and ω_{U} have this intersection in the centre rectangle, so that the others to the left and right, shown shaded, are curve free. Any brace stiffness $>k_{\mathrm{U}}$ will exclude the contained frequency, as will any $<k_{\mathrm{L}}$.

If L and U are on adjacent bracing curves, we could have either $k_{\mathrm{L}} < k_{\mathrm{U}}$ or $k_{\mathrm{U}} < k_{\mathrm{L}}$, shown in Fig. 5b and c, respectively. In both cases, the frequency range contains the asymptote separating the two curves, ensuring that the outer rectangles contain bracing curves, so only the central rectangle is possibly curve-free. For $k_{\mathrm{L}} < k_{\mathrm{U}}$, the central rectangle contains both curves (Fig. 5b), but is free, if $k_{\mathrm{U}} < k_{\mathrm{L}}$ (Fig. 5c) and we have a second way of excluding the unwanted frequency. In these figures, both k_{L} and k_{U} have the same sign. Whether they are both positive, as shown, or both negative, is immaterial (except that negative bracing may be physically meaningless).

If L and U lie on the $(n-1)$th and $(n+1)$th bracing curves, as in Fig. 5d, the frequency range contains both asymptotes of the nth curve, which therefore lies on all three rectangles. Exclusion is impossible.

Excluding a frequency means being in the shaded area of either Fig. 5a or c, described as *open exclusion* and *closed exclusion*, respectively.

3.1. Assessing exclusion

Determining whether a given brace added to a given structure will exclude frequencies from a stated range $[\omega_L, \omega_U]$ is now straightforward. Bracing stiffnesses k_L and k_U are calculated from Eq. (9). These values determine the curves on which L and U lie, and therefore which of the exclusion diagrams applies. If the resulting picture looks like Fig. 5a or c, suitable stiffness will solve the problem, and if it looks like Fig. 5b or d, the unwanted frequency cannot be excluded. These possibilities are summarised in Table 1.

If the structure has several tunable elements, existing or proposed, each one can be considered in the above manner. Part of the solution is to confirm that a single mode is contained in the range $[\omega_L, \omega_U]$, probably through Gauss factorisation, but for simplicity of the argument, a signed Choleski factorisation is used. The diagonal factor has elements of ± 1, and we have

$$\underset{\sim}{\mathbf{K}} = \underset{\sim}{\mathbf{K}}(0,\omega) = \underset{\sim}{\mathbf{L}}\underset{\sim}{\bar{\mathbf{I}}}\underset{\sim}{\mathbf{L}}^{\mathrm{T}}, \quad \underset{\sim}{\bar{\mathbf{I}}} = \lceil \pm 1 \rfloor, \tag{12}$$

and therefore,

$$\underset{\sim}{\mathbf{K}}^{-1} = \underset{\sim}{\mathbf{L}}^{-\mathrm{T}}\underset{\sim}{\bar{\mathbf{I}}}\underset{\sim}{\mathbf{L}}^{-1}. \tag{13}$$

Bracing stiffnesses are now

$$-1/k = \underset{\sim}{\mathbf{g}}\underset{\sim}{\mathbf{K}}^{-1}\underset{\sim}{\mathbf{g}}^{\mathrm{T}} = \left(\underset{\sim}{\mathbf{g}}\underset{\sim}{\mathbf{L}}^{-\mathrm{T}}\right)\underset{\sim}{\bar{\mathbf{I}}}\left(\underset{\sim}{\mathbf{L}}^{-1}\underset{\sim}{\mathbf{g}}^{\mathrm{T}}\right). \tag{14}$$

Returning to the cantilever example, and bracing to exclude frequencies in the range $[14, 24]$, $\underset{\sim}{\mathbf{K}}_L = \underset{\sim}{\mathbf{K}}(0, 14)$ is formed and factored to give

$$\underset{\sim}{\mathbf{L}}_L^{-\mathrm{T}} = \begin{bmatrix} 0.1299 & -2.198 & 2.207 & 0.1304 \\ \cdot & 6.081 & -6.046 & -0.2319 \\ \cdot & \cdot & 0.0119 & 0.0698 \\ \cdot & \cdot & \cdot & 0.1889 \end{bmatrix}, \qquad \underset{\sim}{\bar{\mathbf{I}}}_L = \lceil 1 \; 1 - 1 \; 1 \rfloor, \tag{15}$$

and $\underset{\sim}{\mathbf{K}}_U = \underset{\sim}{\mathbf{K}}(0, 24)$ similarly gives

$$\underset{\sim}{\mathbf{L}}_U^{-\mathrm{T}} = \begin{bmatrix} 0.3019 & 0.2652 & -0.2485 & 0.1647 \\ \cdot & 0.1768 & 1.232 & -0.7262 \\ \cdot & \cdot & 0.1729 & -0.0569 \\ \cdot & \cdot & \cdot & 0.2177 \end{bmatrix}, \qquad \underset{\sim}{\bar{\mathbf{I}}}_U = \lceil -1 \; 1 - 1 \; 1 \rfloor. \tag{16}$$

Table 1
Exclusion criteria

	$k_L < k_U$	$k_U < k_L$
k_L, k_U opposite sign	Open exclusion is possible (Fig. 5a)	Exclusion is not possible (Fig. 5d)
k_L, k_U same sign	Exclusion is not possible (Fig. 5b)	Closed exclusion is possible (Fig. 5c)

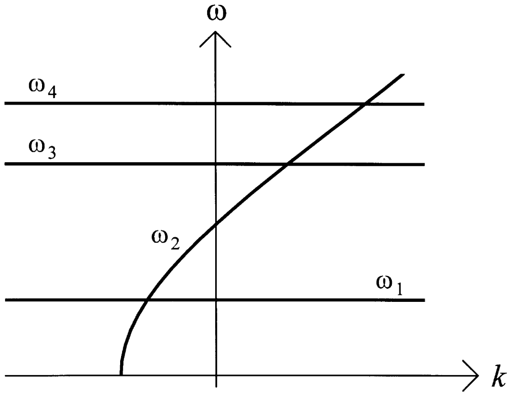

Fig. 6. Bracing graph of a braced single mode.

A selection of different braces can now be readily assessed. Considering bracing of, say, the midspan deflection, $\mathbf{g} = [0 \quad 0 \quad 1 \quad 0]$, $\mathbf{g}\mathbf{L}_{\mathrm{L}}^{-\mathrm{T}} = [\cdot \quad \cdot \quad 0.0119 \quad 0.0698]$ and $k_{\mathrm{L}} = -211.6$, with a similar calculation giving $k_{\mathrm{U}} = 37.5$. All stiffnesses outside these values will give open exclusion. Likewise, bracing the tip rotation will work, with a similar graph, and as shown previously, bracing the tip deflection gives a closed exclusion of the second mode. Bracing the midspan rotation cannot exclude the contained frequency. [3]

3.2. Constructing exclusion

A question that is still open is

> Given bounding frequencies, known to contain 1 natural frequency, is there always a brace capable of excluding the contained frequency?

The answer is 'yes', for the following reason:

The geometric connection $\mathbf{g}$ describes how a brace connects to the freedoms. With a coordinate transformation to the eigenspace of $\mathbf{K} - \omega^2\mathbf{M}$, these become connections to the vibration modes (Barbato and Lawther, 1997), and in this space a brace such as [0..0 1 0..] will connect to one mode only. With such a connection the graph of bracing curves looks like Fig. 6, where the connected mode is the only one to give altered frequencies, and all others are horizontal lines (when the graph is plotted in the eigenvalues k and ω^2, all lines are straight).

Connecting to the included mode produces an open exclusion graph. The brace, in the original coordinates, comes from the inverse of the matrix of eigenvectors, but can be constructed without inversion, or a complete eigensolution, using the orthogonality of the eigenvectors through $\mathbf{K}$ (or $\mathbf{M}$, or $\mathbf{K} - \omega^2\mathbf{M}$): if $\mathbf{u}$ is the vibration mode to be excluded, $\mathbf{g} = \mathbf{u}^{\mathrm{T}}\mathbf{K}$ is the brace that will exclude it. The second mode of the cantilever is $\mathbf{u}_2 = \{-0.2004 \quad 0.9650 \quad 0.1447 \quad 0.0871\}$, giving $\mathbf{g}_2 = \mathbf{u}_2^{\mathrm{T}}\mathbf{K} = [-0.3136 \quad -0.0085 \quad 0.94940.0176]$. This brace geometry can exclude up to the full range of $[\omega_1, \omega_3] = [3.518, 75.16]$.

The above construction requires that the included mode shape is known.

> Can the exclusion be achieved without solving for any eigenvalues or eigenvectors?

[3] Much of the above and following calculation is for demonstration. Explicit formulation of $\mathbf{L}^{-\mathrm{T}}$ is going to destroy any banding of $\mathbf{L}$, and is best avoided in a problem of any size.

Again, the answer is 'yes', but the argument is more complex. It revolves around creating an open exclusion graph, using the knowledge that ω_{L} and ω_{U} lead to mode counts different exactly by 1, and therefore $\bar{\mathbf{I}}_{\mathrm{U}}$ has one more element of -1 than does $\bar{\mathbf{I}}_{\mathrm{L}}$.

From Eq. (14), namely,

$$-1/k = \mathbf{g}\mathbf{K}^{-1}\mathbf{g}^{\mathrm{T}} = \left(\mathbf{g}\mathbf{L}^{-\mathrm{T}}\right)\bar{\mathbf{I}}\left(\mathbf{L}^{-1}\mathbf{g}^{\mathrm{T}}\right) = \mathbf{x}\bar{\mathbf{I}}\mathbf{x}^{\mathrm{T}},$$

$-1/k$ is the length of a vector $\mathbf{x}$ measured through the (pseudo) metric $\bar{\mathbf{I}}$. It is $\sum \pm x_i^2$, with the + corresponding to the positive elements of $\bar{\mathbf{I}}$, and the $-$ corresponding to the negative. The vector $\mathbf{x}$ contains the brace connections in the coordinates of a Choleski transformation, and partitions naturally into $\mathbf{x} = [\mathbf{x}^- \ \mathbf{x}^+]$, giving

$$-1/k = -\sum_i (x^-)_i^2 + \sum_i (x^+)_i^2 \tag{17}$$

in general, with

$$-1/k_{\mathrm{L}} = \mathbf{g}\mathbf{K}_{\mathrm{L}}^{-1}\mathbf{g}^{\mathrm{T}} = \left(\mathbf{g}\mathbf{L}_{\mathrm{L}}^{-\mathrm{T}}\right)\bar{\mathbf{I}}_{\mathrm{L}}\left(\mathbf{L}_{\mathrm{L}}^{-1}\mathbf{g}^{\mathrm{T}}\right) = \mathbf{x}_{\mathrm{L}}\bar{\mathbf{I}}_{\mathrm{L}}\mathbf{x}_{\mathrm{L}}^{\mathrm{T}} = -\sum_i (x_{\mathrm{L}}^-)_i^2 + \sum_i (x_{\mathrm{L}}^+)_i^2 \tag{18}$$

and

$$-1/k_{\mathrm{U}} = \mathbf{g}\mathbf{K}_{\mathrm{U}}^{-1}\mathbf{g}^{\mathrm{T}} = \left(\mathbf{g}\mathbf{L}_{\mathrm{U}}^{-\mathrm{T}}\right)\bar{\mathbf{I}}_{\mathrm{U}}\left(\mathbf{L}_{\mathrm{U}}^{-1}\mathbf{g}^{\mathrm{T}}\right) = \mathbf{x}_{\mathrm{U}}\bar{\mathbf{I}}_{\mathrm{U}}\mathbf{x}_{\mathrm{U}}^{\mathrm{T}} = -\sum_i (x_{\mathrm{U}}^-)_i^2 + \sum_i (x_{\mathrm{U}}^+)_i^2, \tag{19}$$

at the limits of the band.

Both these involve the same brace $\mathbf{g}$, so $\mathbf{x}_{\mathrm{U}}$ and $\mathbf{x}_{\mathrm{L}}$ are obviously related as

$$\mathbf{x}_{\mathrm{L}} = \mathbf{x}_{\mathrm{U}}\mathbf{L}_{\mathrm{U}}^{\mathrm{T}}\mathbf{L}_{\mathrm{L}}^{-\mathrm{T}} = \mathbf{x}_{\mathrm{U}}\mathbf{T}_{\mathrm{UL}}, \tag{20}$$

and when partitioned,

$$\begin{bmatrix}\mathbf{x}_{\mathrm{L}}^- & \mathbf{x}_{\mathrm{L}}^+\end{bmatrix} = \begin{bmatrix}\mathbf{x}_{\mathrm{U}}^- & \mathbf{x}_{\mathrm{U}}^+\end{bmatrix}\begin{bmatrix}\mathbf{T}_{\mathrm{UL}}^{--} & \mathbf{T}_{\mathrm{UL}}^{-+} \\ \mathbf{T}_{\mathrm{UL}}^{+-} & \mathbf{T}_{\mathrm{UL}}^{++}\end{bmatrix}. \tag{21}$$

For open exclusion, k_{U} must be positive, or $-1/k_{\mathrm{U}}$ negative. Choosing $\mathbf{x}_{\mathrm{U}}^+ = 0$ and $\mathbf{x}_{\mathrm{U}}^- \neq 0$ ensures this, and further gives

$$\mathbf{x}_{\mathrm{L}}^- = \mathbf{x}_{\mathrm{U}}^-\mathbf{T}_{\mathrm{UL}}^{--}, \tag{22a}$$

$$\mathbf{x}_{\mathrm{L}}^+ = \mathbf{x}_{\mathrm{U}}^-\mathbf{T}_{\mathrm{UL}}^{-+}. \tag{22b}$$

Open exclusion equally requires k_{L} to be negative, or $-1/k_{\mathrm{L}}$ to be positive, which is guaranteed if $\mathbf{x}_{\mathrm{L}}^- = 0$. An algorithm that produces both $\mathbf{x}_{\mathrm{U}}^+ = 0$ and $\mathbf{x}_{\mathrm{L}}^- = 0$, simultaneously with $\mathbf{x}_{\mathrm{U}}^- \neq 0$ and $\mathbf{x}_{\mathrm{L}}^+ \neq 0$, must construct a brace for open exclusion of the contained mode.

If the frequency to be excluded is the nth then $\mathbf{x}_{\mathrm{U}}^-$ has n elements, and $\mathbf{x}_{\mathrm{L}}^-$ has $n-1$. $\mathbf{T}_{\mathrm{UL}}^{--}$ has one more row than columns, and the rows are therefore linearly dependent. Choosing $\mathbf{x}_{\mathrm{U}}^-$ to be this dependence completes the construction, except for a transformation which writes the brace $\mathbf{g}$ in the original coordinates.

This algorithm is now demonstrated using the matrix factors of Eqs. (15) and (16) to construct a brace for openly excluding the contained second mode from the range [14,24].

The transformation from $\mathbf{x}_{\mathrm{U}}$ to $\mathbf{x}_{\mathrm{L}}$ is

$$\underset{\sim}{\mathbf{x}}_{\mathrm{L}} = \underset{\sim}{\mathbf{x}}_{\mathrm{U}} \underset{\sim}{\mathbf{T}}_{\mathrm{UL}} = \underset{\sim}{\mathbf{x}}_{\mathrm{U}} \begin{bmatrix} 0.4304 & -37.49 & 37.83 & 2.763 \\ 0 & 34.40 & -34.68 & -2.547 \\ 0 & 0 & 0.06872 & 0.6890 \\ 0 & 0 & 0 & 0.8676 \end{bmatrix}. \tag{23}$$

$\bar{\underset{\sim}{\mathbf{I}}}_{\mathrm{U}}$ has its two negative elements in positions 1 and 3, and $\bar{\underset{\sim}{\mathbf{I}}}_{\mathrm{L}}$ has its one negative element in position 3. $\underset{\sim}{\mathbf{T}}_{\mathrm{UL}}^{--}$ relates the elements of $\underset{\sim}{\mathbf{x}}_{\mathrm{L}}$ with negative measure to the similar elements of $\underset{\sim}{\mathbf{x}}_{\mathrm{U}}$, and is therefore the partition of $\underset{\sim}{\mathbf{T}}_{\mathrm{UL}}$ contained in rows 1 and 3, and in column 3:

$$\underset{\sim}{\mathbf{T}}_{\mathrm{UL}}^{--} = \begin{bmatrix} 37.83 \\ 0.06872 \end{bmatrix}. \tag{24}$$

Choosing $\underset{\sim}{\mathbf{x}}_{\mathrm{U}}^{-} = [0.06872 \quad -37.83]$, together with $\underset{\sim}{\mathbf{x}}_{\mathrm{U}}^{+} = 0$ will give a brace for open exclusion. This brace,

$$\underset{\sim}{\mathbf{x}}_{\mathrm{U}} = [0.06872 \quad 0 \quad -37.83 \quad 0] \tag{25}$$

transforms to

$$\underset{\sim}{\mathbf{g}} = \underset{\sim}{\mathbf{x}}_{\mathrm{U}} \underset{\sim}{\mathbf{L}}_{\mathrm{U}} = [-0.0010 \quad 0.0015 \quad 0.9661 \quad 0.2582], \tag{26}$$

after norming to $\underset{\sim}{\mathbf{g}}\underset{\sim}{\mathbf{g}}^{\mathrm{T}} = 1$.

Stiffnesses with these connections are $k_{\mathrm{L}} = -74.4$ and $k_{\mathrm{U}} = 35.0$. As confirmation of the open exclusion, the graph of this brace is shown in Fig. 7.

The condition $\underset{\sim}{\mathbf{x}}_{\mathrm{U}}^{-} \neq 0$, $\underset{\sim}{\mathbf{x}}_{\mathrm{U}}^{+} = 0$, $\underset{\sim}{\mathbf{x}}_{\mathrm{L}}^{-} = 0$, and $\underset{\sim}{\mathbf{x}}_{\mathrm{L}}^{+} \neq 0$ is sufficient but not necessary for open exclusion, which can be achieved in many more ways, though not nearly as simply as in the above construction.

3.3. Constructing closed exclusion

The foregoing section shows constructions for open exclusion of frequencies, both with and without knowledge of the natural frequencies and modes. Closed exclusions differ from open in that they are destroyed if the brace is made too stiff. The offending frequency is cleared from the range, but the too-stiff brace has raised a lower frequency to now occupy the band.

Closed exclusion can exist with both positive and negative brace stiffnesses, and with two exceptions, all frequencies are open to exclusion by either. Closed exclusion of the nth frequency with positive stiffness

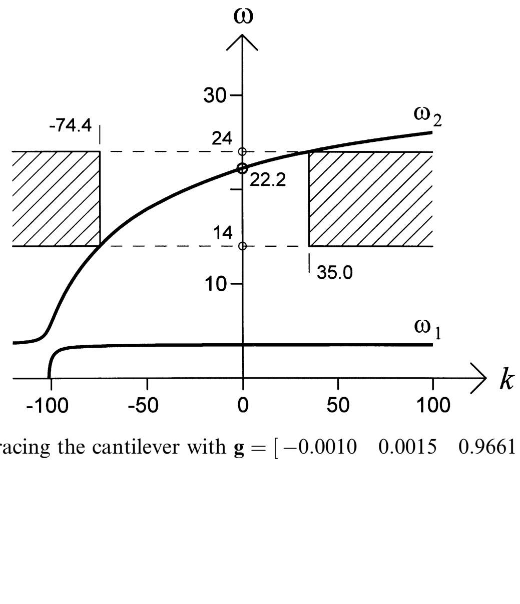

Fig. 7. Bracing the cantilever with $\mathbf{g} = [-0.0010 \quad 0.0015 \quad 0.9661 \quad 0.2582]$.

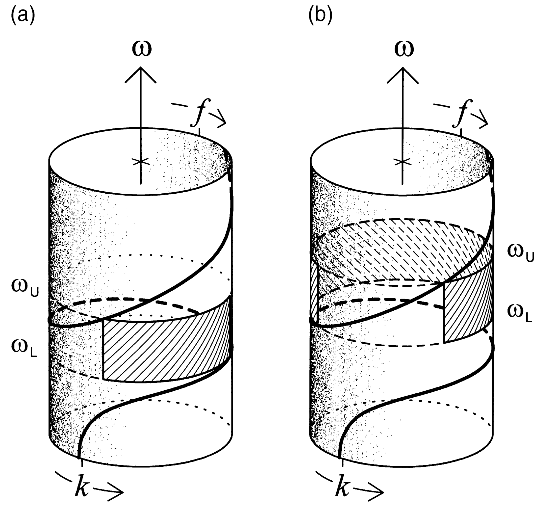

Fig. 8. Fig. 1b with (a) its closed exclusion and (b) its open exclusion, drawn on the cylinder.

involves a graph like Fig. 5c, where U is on the nth bracing curve, and L is on the $(n-1)$th. The fundamental frequency has no lower curve, and cannot be excluded in this way. For similar reasons, the highest frequency has no closed exclusion with negative bracing stiffness. All others are logically possible, and constructible, both with and without knowledge of the frequencies and modes.

The results are not particularly interesting. The algorithms are somewhat more complex than those above, and with open exclusion always possible, there seems no reason to prefer a closed exclusion.

3.4. Closed v open exclusion

Closed and open exclusions are physically distinct. One can be destroyed by a brace being made too stiff, and the other can not. Bracing the included mode only, with a bracing diagram as in Fig. 6, seems a 'natural' way to move an included frequency, and this can only give an open exclusion diagram. In this sense, open exclusion would seem natural and closed exclusion a bit artificial.

In another sense, they are not all that different. Fig. 8a shows a bracing curve drawn on a cylinder, in the manner of Fig. 3d. The band $[\omega_L, \omega_U]$ appears as a hoop, and the hatched region shows a part of this hoop that gives a closed exclusion. A change to the brace geometry means a movement of the helix, forcing the exclusion region to move horizontally. If the movement is such that the new region includes the $f = 0$ axis, diametrically opposite the $k = 0$ axis, the exclusion has opened. A brace can be changed continuously so that open exclusion closes, and vice versa. The changeover occurs when the vertical line, $f = 0$, the horizontal circle, $\omega = \omega_L$ (or ω_U), and the bracing curve are concurrent.

Fig. 8b shows an open exclusion on the same bracing curve, emphasising that the distinction is containment of the f axis (Fig. 8a and b comprise Fig. 1b drawn on the cylinder).

4. Closing remarks

Altering structures to exclude unwanted vibration frequencies has been analysed at the level of a parent problem, in which a single (i.e. rank 1) brace is used to exclude a single frequency. Assessing a given brace has been demonstrated, but more interestingly, it has been shown that the problem is always solvable, with algorithms for workable braces, both with and without detailed knowledge of the mode to be excluded.

A rank 1 brace is equivalent to a truss member in that both are substructures affecting a single generalised freedom. Physically, a truss member can only stress a structure through two equal and opposite forces acting along the same line (or only one force if the member connects to the ground). General rank 1 braces can be more complex than this. Even in the simple example of the cantilever, the physical substructure corresponding to the brace of Eq. (26), used in producing Fig. 7, is far from obvious. Braces have been constructed mathematically, but just how practical it would be to construct one of these physically is another question.

The exclusion problem, with the initial form $(\underset{\sim}{\mathbf{K}} - \omega^2\underset{\sim}{\mathbf{M}})\underset{\sim}{\mathbf{u}} = \underset{\sim}{\mathbf{0}}$, has been treated entirely in terms of changes to $\underset{\sim}{\mathbf{K}}$, and it was earlier noted that an alternative form equally allows changes to $\underset{\sim}{\mathbf{M}}$. $\underset{\sim}{\mathbf{K}}$ is always positive definite for a structure, a centrally important property for mode counting. $\underset{\sim}{\mathbf{M}}$ is likely to be so, too, but not necessarily. If $\underset{\sim}{\mathbf{M}}$ is only semi-definite, mode counting needs closer scrutiny. Also, a rank 1 change to $\underset{\sim}{\mathbf{M}}$ is hard to imagine. Mass is a scalar, and affects all directions equally. A point mass added to a position has the same mass in the x, the y and the z directions. But it is still valid to see this as 3 rank 1 changes, *albeit* simultaneous, and therefore able to be built from the parent problem.

Of course, changes could be made to both $\underset{\sim}{\mathbf{K}}$ and $\underset{\sim}{\mathbf{M}}$, but these will only be rank 1 in the simplest cases.

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Appendix A. Bracing curves in the neighbourhood of a singular $\mathbf{K}(\omega)$

Transformation to modal coordinates $\underset{\sim}{\mathbf{z}} = \underset{\sim}{\mathbf{U}}^{-1}\underset{\sim}{\mathbf{u}}$, where $\underset{\sim}{\mathbf{U}}$ is the matrix of eigenvectors, gives diagonal stiffness and mass matrices. With the columns of $\underset{\sim}{\mathbf{U}}$ scaled so that $\underset{\sim}{\mathbf{U}}^{\mathrm{T}}\underset{\sim}{\mathbf{K}}\underset{\sim}{\mathbf{U}} = \underset{\sim}{\mathbf{I}}$, the corresponding (diagonal) mass matrix is $\underset{\sim}{\mathbf{D}}^{\mathrm{M}} = \underset{\sim}{\mathbf{U}}^{\mathrm{T}}\underset{\sim}{\mathbf{M}}\underset{\sim}{\mathbf{U}}$, brace connections $\underset{\sim}{\mathbf{g}}$ transform to modal connections $\underset{\sim}{\boldsymbol{\gamma}} = \underset{\sim}{\mathbf{g}}\underset{\sim}{\mathbf{U}}$, and Eq. (9) becomes

$$1/k = -\sum_i \frac{\gamma_i^2\omega_i^2}{\omega_i^2 - \omega^2}. \tag{A.1}$$

The connection to mode n is γ_n, and the bracing curve in the neighbourhood of ω_n (where $\underset{\sim}{\mathbf{K}}(\omega)$ is singular) is different depending on whether or not this connection is zero.

(i) If the brace connects to the nth mode then $\gamma_n \neq 0$, $1/k$ becomes infinite at $\omega = \omega_n$ and k is therefore 0, confirming that the bracing curve passes through $(0, \omega_n)$. In the neighbourhood of ω_n, let $\omega^2 = \omega_n^2 + \varepsilon$, when Eq. (A.1) is

$$k \approx \frac{\varepsilon}{\gamma_n^2\omega_n^2}.$$

The bracing curve passes through $(0, \omega_n)$ with a finite slope

$$\frac{\partial\omega}{\partial k} = \frac{\gamma_n^2\omega_n}{2}.$$

At the origin, $k = 0$, the slope is positive, but the origin could be at any k, so,

$$\frac{\partial \omega}{\partial k} > 0 \quad \forall k$$

showing that the bracing curves are monotonic increasing (unless they are the horizontal lines of the next section).

(ii) Eq. (7) is more useful for considering $\gamma_n = 0$. With rank 1 bracing, and a transformation to modal coordinates,

$$\left(\underset{\sim}{\mathbf{I}} + \underset{\sim}{\gamma}^{\mathrm{T}} k \underset{\sim}{\gamma} - \omega^2 \underset{\sim}{\mathbf{D}}^{\mathrm{M}}\right) \underset{\sim}{\mathbf{z}} = \underset{\sim}{\mathbf{0}}. \tag{A.2}$$

Both $\underset{\sim}{\mathbf{I}}$ and $\underset{\sim}{\mathbf{D}}^{\mathrm{M}}$ are diagonal, so at $k = 0$, the equations separate, as they must, because the modal coordinates are derived at $k = 0$. In general, $\underset{\sim}{\gamma}^{\mathrm{T}} \underset{\sim}{\gamma}$ is not diagonal, and the equations fail to separate for other k, but if $\gamma_n = 0$ the nth equation separates from the others to give the eigenvector as $\underset{\sim}{\mathbf{z}} = \underset{\sim}{\mathbf{e}}_n$, and the eigenvalue from

$$1 + 0k - \omega^2/\omega_n^2 = 0,$$

i. e.

$$\omega = \omega_n \quad \forall k.$$

The bracing curve is a horizontal line. If a brace has no connection to a mode, it has no influence on it. The relation,

$$\frac{\partial \omega}{\partial k} = \frac{\gamma_n^2 \omega_n}{2}$$

applies for all γ_n including $\gamma_n = 0$.

Returning to Eq. (A.1), which is now

$$1/k = -\sum_{i \neq n} \frac{\gamma_i^2 \omega_i^2}{\omega_i^2 - \omega^2},$$

when $\gamma_n = 0$, the nth mode ω_n has no effect on other bracing curves in its neighbourhood, or anywhere else. Other curves may intersect its horizontal bracing curve, or asymptote to it. It can almost completely be ignored. Almost, but not quite. It does act as a boundary for adjacent mode count regions (Section 2.3), and so gives bounds on how far the brace can move the frequencies.

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